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THE LAW OF INFINITE CARDINAL ADDITION IS WEAKER THAN THE AXIOM OF CHOICE

BY

J. D. HALPERN AND PAUL E. HOWARD

Dedicated to the memory of Andrzej Mostowski

ABSTRACT. We construct a permutation model of set theory with urelements in which C_2 (the choice principle restricted to families whose elements are unordered pairs) is false but the principle, "For every infinite cardinal m, $2m = m$" is true. This answers in the negative a question of Tarski posed in 1924. We note in passing that the choice principle restricted to well-ordered families of finite sets is also true in the model.

1. The axiom of choice (AC) implies that cardinal arithmetic is trivial for infinite cardinals. That is:

$$m + n = \sup(m, n) \quad \text{if one of } m, n \text{ is infinite}$$

and

$$m \cdot n = \max(m, n) \quad \text{if one of } m, n \text{ is infinite and the other is different from 0.}$$

Of course AC implies that any two cardinals are comparable so $\sup(m, n) = \max(m, n)$, but each of the laws as stated above follow respectively from the following laws without additional appeal to AC.

(1) $m + m = m$ for all infinite cardinals m.

(2) $m \cdot m = m$ for all infinite cardinals m.

In 1924 Tarski [3] gave a proof of (AC) from (2) and asked [4] if (AC) is provable from (1). We give a negative answer to this question by exhibiting a permutation model of ZFU (Zermelo-Fraenkel set theory modified so as to allow urelements) in which C_2 (the axiom of choice for families whose elements are pairs) is false but (1) is true.(1)

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(1) The proof of our result was completed near the end of December, 1972. We learned subsequently that G. Sageev (Notices Amer. Math. Soc. **20** (1973), A22) had already shown that the answer to Tarski's question was negative even in the context of set theory with regularity.

This paper was presented to the Association for Symbolic Logic in January 1973, to the Michigan-Ohio Logic Seminar in May and June 1973, to the International Colloquium in Infinite and Finite Sets at Keszthely, Hungary in June 1973. It was also presented as a Research Announcement in Bull. Amer. Math. Soc. **80** (1974), under a different title [1].

A permutation model (Specker [2]) is determined by a set of points(2) (i.e. urelements) U, a group G of permutations of U, and a conjugated filter J of subgroups of G.

Let U be a countable set of points. G and J will be described in terms of an arbitrary 1-1 correspondence between U and $\omega^{(\omega)}$ where $\omega^{(\omega)}$ is the set of all $s \in {}^{\omega}\omega$ which are eventually zero, i.e., $(\exists n)(\forall j > n)s_j = 0$. For ease of exposition, in the sequel we will not distinguish between $s \in \omega^{(\omega)}$ and the element in U to which it corresponds. For $s \in U$, let the *pseudo-length* of s be defined as the least number k such that $\forall l \geqslant k$, $s_l = 0$. Call a permutation φ of U *bounded* if the set of those elements moved by φ is of bounded pseudo-length i.e. for some k, $\varphi(s) \neq s$ implies the pseudo-length of s is less than k. Let G be the group of all bounded permutations of U. For $n \in \omega$, $s \in U$ let $n \mid s$ (read "s after n") $= (s_n, s_{n+1}, \ldots)$ and let A^n_s (read "the n-block containing s") $= \{t \in U\colon n \mid t = n \mid s\}$ except that we will let A^n be the n-block containing $\mathbf{0}$, the sequence which is identically zero. Note the following: (1) If s has pseudo-length n, then $n \mid s = \mathbf{0}$. (2) $\varphi \in G$ implies that for some k, the set of those points moved by φ (this set is usually called the support of φ but we will be using "support" in a different way) is included in A^k. Also note that $s \in U$ is uniquely determined (for each $n \in \omega$) by $s \mid n$ and $n \mid s$. We will call $s \mid n$ the n-location of s. For each $n \in \omega$ let $G_n = \{\varphi \in G\colon 1 \text{ and } 2 \text{ and } 3\}$ where

1. φ fixes the n-block containing $\mathbf{0}$, i.e., A^n, pointwise.
2. φ fixes $\{A^n_s\colon s \in U\}$, i.e., $n \mid s = n \mid t$ implies $n \mid \varphi(s) = n \mid \varphi(t)$.
3. φ preserves n-locations, i.e., $s \mid n = \varphi(s) \mid n$.

Note that $G_0 = \{\varphi \in G\colon \varphi(\mathbf{0}) = \mathbf{0}\}$. Also note that if f is a 1-1 partial function on U into U, having an extension in G satisfying 1, 2, and 3 then f has an extension in G_n. Let $J =$ the filter of subgroups generated by the G_n, $n \in \omega$.

LEMMA 1.1. *J is a proper filter and closed under conjugation.*

PROOF. J is proper since $n < m$ implies $\{1_G\} \neq G_m \subseteq G_n$. J is closed under conjugation since if $\{a\colon \varphi(a) \neq a\}$ is included in A^k (and $k \geqslant n$) then $G_k \subseteq \varphi^{-1} G_n \varphi$.

Briefly the description of M is as follows: Let R be the function on ordinals defined recursively by $R(0) = U$ and $R(\alpha) = P(\bigcup_{\beta<\alpha} R_\beta)$ for $\alpha > 0$. Then $V = \bigcup_{\alpha \in \mathrm{ord}} R(\alpha)$ is called the set universe over U. Any permutation φ of U extends uniquely to an automorphism (with respect to $\in$) φ^* of V.

(2) Permutation models can also be used for set theories whose axioms are those of ZF except that the axiom of regularity is weakened or eliminated. Specker deals with such a theory but his development of permutation models carries over verbatim for ZFU. We use ZFU rather than such a theory because it seems more natural.

We will confuse φ^* with φ in the future. In the style of Lévy call an $x \in V$, *J-symmetric*, if there exists $H \in J$ such that H fixes x, i.e., $\varphi \in H \Rightarrow \varphi(x) = x$. Then M is just the substructure of V consisting of all those sets x such that x and every element of the transitive closure of x is J-symmetric.

LEMMA 1.2. *If $A \subseteq U$ and $A \in M$ then $M \models$ (A is countable) (i.e. A is M-countable) iff $\exists k \in \omega$, such that $A \subseteq A^k$.*

PROOF. If $M \models$ (A is countable), then some G_k fixes a 1-1 correspondence between A and ω. Hence G_k fixes A pointwise and therefore $A \subseteq A^k$.

If $A \subseteq A^k$, then any 1-1 correspondence between A and ω is fixed by G_k. Hence A is M-countable.

THEOREM 1. *C_2 is false in M.*

PROOF. The set of all (unordered) pairs of elements of U is in M. However for any k there is a $\varphi \in G_k$ such that φ interchanges two elements of U. Hence there is no J-symmetric choice set for this set of pairs.

It remains to show that (1) holds in the structure M. To this end consider a set $y \in M$ and take k_0 such that G_{k_0-1} fixes y and $k_0 > 0$. Let:

$y_1 = \{x \in y$: G_{k_0} fixes $x\}$,

$y_2 = \{x \in y$: G_{k_0} does not fix x and there exists an M-countable $A \subseteq U$ such that $\forall \varphi \in G_{k_0}$ (φ fixes A pointwise implies $\varphi(x) = x$)$\}$,

$y_3 = \{x \in y$: for all M-countable $A \subseteq U$, $\exists\, \varphi \in G_{k_0}$ such that φ fixes A pointwise and $\varphi(x) \neq x\}$.

Then y is the disjoint union of y_1, y_2, and y_3. Furthermore $y_1, y_2, y_3 \in M$ since G_{k_0} fixes each of them. (Actually G_{k_0-1} fixes y_3. We will use this fact in §3.) y_1 is well orderable in M since G_{k_0} fixes each element and hence a well ordering of y_1. We will show that for $z = y_2$ or $z = y_3$ there is a 1-1 function F with domain $2 \times z$ and range z such that G_{k_0+1} fixes F and hence $F \in M$. We will call such an F a $2m$-ing of z. This shows that every cardinal in M is the sum of three cardinals one of which is well orderable and the other two satisfy the equation $2m = m$ and hence that (1) is true in the structure M.

(*Note.* If y is finite, it follows that y_2 and y_3 must be empty and from this it follows that any well-ordered collection of finite sets in M has a choice function in M.)

2. y_2 **is** $2m$**-able.**

DEFINITION 2.1. An M-countable $A \subseteq U$ is called an n-support of $x \in M$ if

(a) $\forall \varphi \in G_n$, φ fixes A pointwise $\Rightarrow \varphi(x) = x$.

(b) $\forall s \in U$, $A^n_s \cap A = \varnothing$ or $A^n_s \subseteq A$.

(c) $A^n \subseteq A$.

((a) is the crucial property; (b) and (c) are added for ease of exposition.)

LEMMA 2.2. *There are two functions*

$$f_0, f_1 \colon U - A^{k_0} \xrightarrow[\text{into}]{\text{1-1}} U - A^{k_0}$$

such that

(a) f_0, f_1 *have disjoint ranges.*

(b) f_0, f_1 *are each fixed by* G_{k_0+1} *and thus* $f_0, f_1 \in M$.

(c) f_0, f_1 *preserve* k_0*-locations and each maps* k_0*-blocks to* k_0*-blocks; so the restrictions of* f_0, f_1 *to an M-countable set have extensions in* G_{k_0}.

PROOF. Let $f_0(s) = s'$ where s' is like s except that $s'_{k_0} = 2s_{k_0}$. Let $f_1(s) = s''$ where s'' is like s except that

$$s''_{k_0} = 2s_{k_0} + 1 \quad \text{if } s \notin A^{k_0+1},$$

$$s''_{k_0} = 2s_{k_0} - 1 \quad \text{if } s \in A^{k_0+1} - A^{k_0}.$$

It is clear that

$$f_0, f_1 \colon U - A^{k_0} \xrightarrow[\text{into}]{\text{1-1}} U - A^{k_0}.$$

Parts (a) and (c) are also clear. We prove that G_{k_0+1} fixes f_0. The proof that G_{k_0+1} fixes f_1 is almost identical.

Suppose $\varphi \in G_{k_0+1}$, then

$$(*) \qquad \phi(\langle s, f_0(s)\rangle) = \langle \phi(s), \phi(f_0(s))\rangle.$$

We now compute

$$[\phi(f_0(s))]_i = \begin{cases} \phi(s)_i & \text{if } i > k_0 \text{ since } \phi \in G_{k_0+1} \text{ and } (k_0+1)|s = (k_0+1)|f_0(s), \\ 2s_i & \text{if } i = k_0 \\ s_i & \text{if } i < k_0 \end{cases} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{since } \phi \in G_{k_0+1}$$

and

$$[f_0(\phi(s))]_i = \begin{cases} \phi(s)_i & \text{if } i > k_0, \\ 2[\phi(s)]_i = 2s_i & \text{if } i = k_0, \\ \phi(s)_i = s_i & \text{if } i < k_0. \end{cases}$$

Therefore $\varphi(f_0(s)) = f_0(\varphi(s))$. So rewriting $(*)$

$$\phi(\langle s, f_0(s)\rangle) = \langle \phi(s), f_0(\phi(s))\rangle \in f_0.$$

Since G_{k_0+1} is a group the above computation also holds for φ^{-1}. Hence φ fixes f_0.

LEMMA 2.3 (SUPPORT LEMMA). *If A is a k_0-support of x, and $\varphi_1, \varphi_2 \in G_{k_0}$ and $\varphi_1|A = \varphi_2|A$, then $\varphi_1(x) = \varphi_2(x)$.*

This lemma follows from the facts that $\varphi_2^{-1}\varphi_1 \in G_{k_0}$ and $\varphi_2^{-1}\varphi_1$ fixes A pointwise.

Suppose A is a k_0-support of $x \in y_2$. Then $f_0 | A$, $(f_1 | A)$, has an extension in G_{k_0}. The extension is not unique but the following corollary which is a direct consequence of the support lemma gives the desired amount of uniqueness.

COROLLARY 2.3. *If φ and φ' both extend $f_0 | A$, $(f_1 | A)$, where A is a k_0-support of x and $\varphi, \varphi' \in G_{k_0}$, then $\varphi(x) = \varphi'(x)$.*

LEMMA 2.4. *If A and A' are k_0-supports of x then so is $A \cap A'$.*

PROOF. We need only show that if $\varphi \in G_{k_0}$ and φ fixes $A \cap A'$ pointwise then $\varphi(x) = x$. We show that φ can be represented as a finite sequence of elements of G_{k_0} such that each element of the sequence fixes A pointwise or fixes A' pointwise. Essentially such a representation is possible because A and A' consist of k_0-blocks and G_{k_0} is M-countably transitive on the set of k_0-blocks.

Choose an integer m such that $m > k_0 + 1$ and A^m has the following properties:

1. $(\forall s \in U - A^m)\ (\varphi(s) = s)$,
2. $A \cup A' \subseteq A^m$.

m exists because φ is bounded, A and A' are supports, and for $m \leqslant n$, $A^m \subseteq A^n$. Let B be any m-block different from A^m. (For definiteness one might take $B = A_t^m$ where t is the sequence defined by $t_m = 1$ and $t_j = 0$ for $j \neq m$.) Let η_1 be the element of G_{k_0} which interchanges each element $s \in A' - A$ with the element $r \in B$ having the same m-location as s (i.e. $s | m = r | m$). η_1 is the identity elsewhere. Let η_2 be the element of G_{k_0} which interchanges each element $s \in A^m - A'$ with the element $r \in B$ having the same m-location as s. η_2 is the identity elsewhere. Let φ_1 be the element of G_{k_0} which is identical to φ except that φ_1 acts on B instead of A^m, that is $\varphi_1 = (\eta_2\eta_1)\varphi(\eta_2\eta_1)$. Then η_1 fixes A pointwise, η_2 fixes A' pointwise and φ_1 fixes $A \cup A'$ pointwise. Clearly $\varphi = \eta_1\eta_2\varphi_1\eta_1\eta_2$. Hence the proof is complete.

COROLLARY 2.5. *If A and A' are k_0-supports of $x \in y_2$ then $A \cap A' \not\subseteq A^{k_0}$.*

PROOF. If $A \cap A' \subseteq A^{k_0}$, then A^{k_0} is a support of x by Lemma 2.4 and hence $\varphi \in G_{k_0} \Rightarrow \varphi(x) = x$ contradicting $x \in y_2$. The following lemma is a direct consequence of 2.3 and 2.4.

LEMMA 2.6. *If A and A' are k_0-supports of k, $\varphi \supseteq f_0 | A$, $\varphi' \supseteq f_0 | A'$ and $\varphi, \varphi' \in G_{k_0}$ then $\varphi(x) = \varphi'(x)$. (Similarly for f_1.)*

DEFINITION 2.7. For each $x \in y_2$ let x_0, (x_1), be the unique member of

$\{\varphi(x)\colon \exists A, A$ is a k_0-support of x, $\varphi \in G_{k_0}$ and $\varphi \supseteq f_0|A$, $(\varphi \supseteq f_1|A)\}$.

Note. $x_0, x_1 \in y_2$ since G_{k_0} fixes y_2.

THEOREM 2. *The function F defined by $F(0, x) = x_0$, $F(1, x) = x_1$ is a* 1-1 *function from* $2 \times y_2$ *into* y_2 *and is in M (fixed by G_{k_0+1}).*

PROOF. $F \in M$ since the group $G \in M$ and hence the description of x_0 and x_1 in terms of x can be carried out in M. This uses the fact that M is a model of ZFU. It remains to prove that F is 1-1. Let $x, z \in y_2$ and let A be a k_0-support of both x and z. (The union of a k_0-support of x with a k_0-support of z is a k_0-support of both.) Let $\varphi, \varphi_1 \in G_{k_0}$ such that $\varphi \supseteq f_0|A$ and $\varphi_1 \supseteq f_1|A$. Since φ and φ_1 are automorphisms of M we have $x \neq z \Rightarrow \varphi(x) \neq \varphi(z)$ and $\varphi_1(x) \neq \varphi_1(z)$. Hence $F(0, x) \neq F(0, z)$ and $F(1, x) \neq F(1, z)$. It remains to show that $F(0, x) \neq F(1, z)$ i.e. $x_0 \neq z_1$. Well, x_0 and $z_1 \in y_2$, $\varphi(A)$ is a support of x_0 and $\varphi_1(A)$ is a support of z_1. But $\varphi(A) \cap \varphi_1(A) = A^{k_0}$. So the equality of x_0 and z_1 contradicts Corollary 2.5. Q.E.D.

3. y_3 is $2m$-able. We will prove the following theorem:

THEOREM 3. *For every $x \in y_3$ there is an infinite set $D_x \in M$ such that*:

(1) $D_x \subseteq y_3$,

(2) $\forall \varphi \in G_{k_0+1}, \forall z_1, z_2 \in D_x [\varphi(z_1) = z_1$ *iff* $\varphi(z_2) = z_2$ *and* $(z_1 \neq z_2 \Rightarrow \varphi(z_1) \neq z_2)]$,

(3) $x \in D_x$.

Before proving the theorem we show why the theorem is sufficient to obtain the $2m$-ing of y_3 in M.

For each $x \in y_3$, let

$$M_x = \{y\colon \forall \phi \in G_{k_0+1}, \phi(y) = y \text{ iff } \phi(x) = x\}$$

and let $\mathcal{D}_x = \{\varphi(y)\colon y \in M_x \;\&\; \varphi \in G_{k_0+1}\}$.

LEMMA 3.1. *$\{\mathcal{D}_x\colon x \in y_3\}$ is a partition of y_3 each element of which is invariant under G_{k_0+1}.*

The proof is an easy calculation. It suffices to obtain for each x, a $2m$-ing of $\mathcal{D}_x$ invariant under G_{k_0+1}.

We define a transitivity class (relative to G_{k_0+1}) of $\mathcal{D}_x$ to be a subset z of $\mathcal{D}_x$ such that for each $y \in z$, $z = \{\varphi(y)\colon \varphi \in G_{k_0+1}\}$.

Let $x_0 \in y_3$ and choose $D_{x_0} \subseteq M_{x_0}$ such that D_{x_0} intersects each transitivity class of $\mathcal{D}_{x_0}$ in one element. Any subset of M_{x_0} is in the model M since the element of J which fixes x_0 also fixes each element of M_{x_0}. Hence $D_{x_0} \in M$. Then $\{\varphi(D_{x_0})\colon \varphi \in G_{k_0+1}\}$ is a partition of $\mathcal{D}_{x_0}$. D_{x_0} is infinite by Theorem 3 and well orderable in the model M since it is a subset of M_{x_0} which is well orderable in M. (Any well ordering of M_{x_0} is in the model M.) Hence there is a $2m$-ing, say F, of D_{x_0} in the model M. Then for each $\varphi \in G_{k_0+1}$, $\varphi(F)$ is a $2m$-ing of $\varphi(D_{x_0})$ since φ is an automorphism of the model M. Hence $\bigcup\{\varphi(F)$: $\varphi \in G_{k_0+1}\}$ is a $2m$-ing of $\mathcal{D}_{x_0}$, is in M, and is invariant under G_{k_0+1}.

DEFINITION 3.2. For any $\varphi \in G$, let $\mathrm{St}(\varphi)$ (read "the stand of φ") $= \{a \in U$: $\varphi(a) \neq a\}$.

First we note that G_{k_0-1} fixes y_3. This follows directly from the following lemma.

LEMMA 3.3. *For all $\varphi \in G$, $z \in M$, if z has an n-support then $\varphi(z)$ has an n-support.*

PROOF. Let B be an n-support of z and let

$$B' = B \cup \bigcup\{A\colon A \text{ is an } n\text{-block and } A \cap \mathrm{St}(\varphi) \neq 0\}.$$

B' is M-countable since $\mathrm{St}(\varphi)$ is M-countable. Furthermore if $\psi \in G_n$ and leaves B' pointwise fixed then $\psi\varphi = \varphi\psi$. So $\psi\varphi(x) = \varphi(x)$. B' obviously satisfies the other requirements of an n-support of $\varphi(x)$.

Let $x \in y_3$ and let k be the least number such that x has k-support. $k > k_0$ since $x \in y_3$.

LEMMA 3.4. *Let B be a k-support of x. If A is an M-countable subset of U, A is the union of infinitely many k-blocks, and A is disjoint from B, then there exists $\varphi \in G_{k-1}$ such that* $\mathrm{St}(\varphi) \subseteq A$ *and $\varphi(x) \neq x$.*

PROOF. Since B is not a $(k-1)$-support of x, there exists $\varphi_1 \in G_{k-1}$ such that φ_1 fixes B pointwise and $\varphi_1(x) \neq x$. Let $A_1 = \{C$: C is a k-block and φ_1 moves some element of $C\}$. $\bigcup A_1$ is M-countable and disjoint from B, so there exists $\psi \in G_k$ such that ψ interchanges A_1 with the set of k-blocks of A (or a finite subset of the k-blocks of A if A_1 is finite) and ψ fixes B pointwise. Then $\varphi = \psi\varphi_1\psi^{-1}$ has the desired properties.

DEFINITION 3.5. Let φ be a 1-1 function with domain and range included in U. We say that φ maps n-blocks to k_0-blocks uniformly if its domain is the union of n-blocks, its range is the union of k_0-blocks for any $s, s' \in \mathrm{Domain}(\varphi)$

(i) $s \restriction n = s' \restriction n$ implies $\varphi(s) \restriction k_0 = \varphi(s') \restriction k_0$ and

(ii) $n \restriction s = n \restriction s'$ implies $k_0 \restriction \varphi(s) = k_0 \restriction \varphi(s')$.

Note. If φ maps n-blocks to k_0-blocks uniformly and $\psi \in G_n$ then $\varphi\psi\varphi^{-1}$ has an extension in G_{k_0}. Similarly, if $\psi_1 \in G_{k_0}$, then $\varphi^{-1}\psi_1\varphi$ has an extension in G_n. Replacing "implies" by "iff" in the definition does not change its strength.

The following lemma is a consequence of the fact that $k, (k-1) \geqslant k_0$.

LEMMA 3.6. *If B and C are mutually disjoint M-countable subsets of $U - A^{k_0 - 1}$, B is the union of infinitely many k-blocks ($(k-1)$-blocks), C is the union of infinitely many k_0-blocks, then there exists*

$$\varphi: B \xrightarrow[\text{onto}]{\text{1-1}} C$$

such that φ maps k-blocks ($(k-1)$-blocks) to k_0-blocks uniformly, and φ has an extension in $G_{k_0 - 1}$.

PROOF. (We prove the lemma for k.) For each $n \in \omega$, let A_n be the set of n-locations, i.e.

$$\mathsf{A}_n = \{s | n: n \in \omega\} = \{s: s \text{ is a sequence of length } n\}.$$

To prove the lemma it suffices to find a one-to-one function g from A_k onto A_{k_0} with these two properties: For all $s, s' \in \mathsf{A}_k$

$$s | k_0 - 1 = g(s) | k_0 - 1$$

and

$$(k_0 - 1) | s = (k_0 - 1) | s' \Rightarrow (k_0 - 1) | g(s) = (k_0 - 1) | g(s').$$

But g can be defined by

$$\begin{aligned} g(s_0, s_1, \ldots, s_{k_0-2}, s_{k_0-1}, \ldots, s_{k-1}) \\ = (s_0, s_1, \ldots, s_{k_0-2}, P_{k-k_0+1}(s_{k_0-1}, \ldots, s_{k-1})) \end{aligned}$$

where P_{k-k_0+1} is a 1-1 function from ω^{k-k_0+1} onto ω.

In the sequel, let B be a fixed k-support of x, let F be a $(k+1)$-block disjoint from B and let $H = \{\varphi \in G_k: \varphi \text{ is the identity on } B\}$.

DEFINITION 3.7. Let

$$\delta_1: F \cup A^{k_0+1} \xrightarrow[\text{onto}]{\text{1-1}} A^{k_0+1}$$

such that

(i) δ_1 has an extension in G_{k_0-1},

(ii) δ_1 maps F 1-1 and onto $A^{k_0+1} - A^{k_0}$,

(iii) $\delta_1 | F$ maps k-blocks to k_0-blocks uniformly.

Let δ_2 be a 1-1 function from $U - (F \cup A^{k_0+1})$ onto $U - A^{k_0+1}$ such that

(iv) $\delta_2 | \mathrm{St}(\delta_2)$ has an extension in G_k,

(v) $\mathrm{St}(\delta_2)$ is the union of k-blocks and is disjoint from B.

Let $\Delta = \delta_1 \cup \delta_2$. The existence of δ_1 follows from Lemma 3.6. To get δ_2, we note that $F \cap B = \varnothing$ and $A^k \subseteq B$, therefore $A^k - A^{k_0+1} \subseteq U - (F \cup A^{k_0+1})$ (= domain δ_2). So δ_2 can be (and in fact must be) chosen to be the identity on $A^k - A^{k_0+1}$. It then remains to define δ_2 on $U - (F \cup A^k)$, but this set and $U - A^k$ are both unions of k-blocks and have M countable complements so the existence of δ_2 is clear. Δ is not uniquely determined by the definition but we assume it is fixed in the sequel.

DEFINITION 3.8. Let σ_1 be a 1-1 function from $F \cup A^{k_0+1}$ onto A^{k_0+1} such that

(i) σ_1 has an extension in G_{k_0-1},

(ii) $\sigma_1 | A^{k_0+1}$ has an extension in G_{k_0},

(iii) $\sigma_1 | F$ maps $(k-1)$-blocks to k_0-blocks uniformly.

Let $\sigma = \sigma_1 \cup \sigma_2$. Again the existence of σ_1 follows from Lemma 3.6 mainly. σ is not uniquely determined by the definition but we assume it is fixed in the sequel.

The main difference between σ and Δ can be seen by considering the set $E = \{A_s^{k_0} : A_s^{k_0} \subseteq A^{k_0+1} \;\&\; s \neq 0\}$. Under Δ each element of E is the image of some k-block which is a subset of F. On the other hand under σ some elements of E are images of $(k-1)$-blocks (which are subsets of F) while the remaining elements of E are images of elements of E.

Note that $\sigma, \Delta \in G_{k_0-1}$. Let

$$D = \{\Delta(x), \sigma\Delta(x), \ldots, \sigma^{(n)}\Delta(x), \ldots\}_{n\in\omega}.$$

Then $D \subseteq y_3$.

LEMMA 3.9. *For all $\varphi \in G$, if $\mathrm{St}(\varphi)$ is disjoint from A^{k_0+1} then $\varphi\Delta(x) \neq \sigma^{(n)}\Delta(x)$ for all $n \geq 1$.*

PROOF. Lemma 3.4 and Definition 3.7 (iv) and (v) assure the existence of $\psi \in G_{k-1}$ such that $\mathrm{St}(\psi) \subseteq \Delta^{-1}(F)$ and $\psi(x) \neq x$. Let $\Omega = (\sigma^{(n)}\Delta)\psi(\sigma^{(n)}\Delta)^{-1}$. Then $\Omega\sigma^{(n)}\Delta(x) \neq \sigma^{(n)}\Delta(x)$. On the other hand, $\mathrm{St}(\Omega) \subseteq A^{k_0+1}$ and 3.7 (iv) and (v) together with 3.8 (ii) and (iii) assure that $\Omega \in G_{k_0}$. This fact together with 3.7 (ii) and (iii) assures that $\Delta^{-1}\Omega\Delta \in H$. Hence $\Delta^{-1}\varphi^{-1}\Omega\varphi\Delta \in H$, i.e. $\Omega\varphi\Delta(x) = \varphi\Delta(x)$. Thus Ω distinguishes $\varphi\Delta(x)$ from $\sigma^{(n)}\Delta(x)$.

COROLLARY 3.10. *If $n \neq m$, $\sigma^{(n)}\Delta(x) \neq \sigma^{(m)}\Delta(x)$. Thus D is infinite.*

COROLLARY 3.11. *If G_{k_0+1} and $n < m$ then $\varphi\sigma^{(n)}\Delta(x) \neq \sigma^{(m)}\Delta(x)$.*

PROOF. $\mathrm{St}(\sigma^{(-n)}\varphi\sigma^{(n)})$ is disjoint from A^{k_0+1}.

LEMMA 3.12. *Let $\varphi \in G_{k_0+1}$, $n \in \omega$. Then $\varphi(x) = x$ iff $\varphi(\sigma^{(n)}\Delta(x)) = \sigma^{(n)}\Delta(x)$.*

PROOF. It follows mainly from Definition 3.7 (iv) and (v) that there is $\psi \in H$ such that $\mathrm{St}(\varphi)$ is disjoint from $\mathrm{St}(\psi^{-1}\sigma^{(n)}\Delta\psi)$. Furthermore $\psi^{-1}\sigma^n\Delta\psi(x) = \psi^{-1}\sigma^n\Delta(x)$ since $\psi \in H$. Also $(\sigma^n\Delta)^{-1}\psi^{-1}\sigma^n\Delta \in H$ because of

Definition 3.7 (iv) and (v). Thus $\psi^{-1}\sigma^{(n)}\Delta\psi(x) = \sigma^{(n)}\Delta(x)$, and φ commutes with $\psi^{-1}\sigma^{(n)}\Delta\psi$. This is sufficient to give the desired result.

PROOF OF THEOREM 3. It follows from 3.9 thru 3.12 that D has all of the properties desired of D_x except, perhaps, the property of containing x as a member. If there is a $\varphi \in G_{k_0+1}$ such that $\varphi(x) \in D$, let $D_x = \varphi^{-1}(D)$. Otherwise let $D_x = D \cup \{x\}$. In either case the crucial properties of D carry over to D_x.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA-BIRMINGHAM, BIRMINGHAM, ALABAMA 35294

DEPARTMENT OF MATHEMATICS, EASTERN MICHIGAN UNIVERSITY, YPSILANTI, MICHIGAN 48197